

where $\mathbf{r} = (x, y) \in \mathbb{R}^2$ and the last equation was derived by making use of Eq. (15). Assuming $\hat{a}_{m,n,0} = \hat{a}_{m,n,K_{\text{stk}}+1} = 0$, $m = 1, \dots, K_{\text{row}}$, $n = 1, \dots, K_{\text{col}}$, it is straightforward to verify that

$$T2 = \sum_{p=1}^{K_{\text{stk}}+1} (\hat{a}_{m,n,p} - \hat{a}_{m,n,p-1}) \beta^{d-1} \left(\frac{z_0}{\Delta z_0} - p + \frac{1}{2} \right). \quad (24)$$

By substituting the above expression into Eq. (23), for $k = 1, \dots, K_{\text{pix}}$, we have

$$G_{\theta}(k) = \frac{N}{M^2} \sum_{m=1}^{K_{\text{row}}} \sum_{n=1}^{K_{\text{col}}} \sum_{p=1}^{K_{\text{stk}}+1} \frac{\tilde{a}_{m,n,p}^{z_0} - \tilde{a}_{m,n,p-1}^{z_0}}{\Delta z_0} \beta^{d-1} \left(\frac{z_0}{\Delta z_0} - p + \frac{1}{2} \right) \\ \times \int_{C_k} \beta^d \left(\frac{x}{M} - x_0 - n \right) \beta^d \left(\frac{y}{M} - y_0 - m \right) d\mathbf{r}, \quad (25)$$

where $\mathbf{r} = (x, y) \in \mathbb{R}^2$, $\tilde{a}_{m,n,0}^{z_0} = \tilde{a}_{m,n,K_{\text{stk}}+1}^{z_0} = 0$, for all $m = 1, \dots, K_{\text{row}}$, and $n = 1, \dots, K_{\text{col}}$.

By making use of Eq. (13), the second term on the right-hand side of Eq. (22) can be simplified as follows

$$\frac{1}{C(z_0)} \frac{\partial C(z_0)}{\partial z_0} = \frac{\Delta x_0 \Delta y_0}{C(z_0)} \sum_{m,n,p} \hat{a}_{m,n,p} \frac{\partial}{\partial z_0} \beta^d \left(\frac{z_0}{\Delta z_0} - p \right) \\ = \Delta x_0 \Delta y_0 \sum_{m,n,p} \tilde{a}_{m,n,p}^{z_0} \frac{\partial}{\partial z_0} \beta^d \left(\frac{z_0}{\Delta z_0} - p \right) \\ = \frac{\Delta x_0 \Delta y_0}{\Delta z_0} \sum_{m=1}^{K_{\text{row}}} \sum_{n=1}^{K_{\text{col}}} \sum_{p=1}^{K_{\text{stk}}} \tilde{a}_{m,n,p}^{z_0} \left(\beta^{d-1} \left(\frac{z_0}{\Delta z_0} - p + \frac{1}{2} \right) - \beta^{d-1} \left(\frac{z_0}{\Delta z_0} - p - \frac{1}{2} \right) \right) \\ = \frac{\Delta x_0 \Delta y_0}{\Delta z_0} \sum_{m=1}^{K_{\text{row}}} \sum_{n=1}^{K_{\text{col}}} \sum_{p=1}^{K_{\text{stk}}+1} (\tilde{a}_{m,n,p}^{z_0} - \tilde{a}_{m,n,p-1}^{z_0}) \beta^{d-1} \left(\frac{z_0}{\Delta z_0} - p + \frac{1}{2} \right), \quad z_0 \in \mathbb{R},$$

where we have made use of Eq. (15) and assumed $\tilde{a}_{m,n,0}^{z_0} = \tilde{a}_{m,n,K_{\text{stk}}+1}^{z_0} = 0$, for all $m = 1, \dots, K_{\text{row}}$, and $n = 1, \dots, K_{\text{col}}$. The last identity was derived in a similar way as Eq. (24). The result follows immediately by substituting Eq. (25) and the above equation into Eq. (22).

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